ABOUT THE NATURE OF MANIPULATIVES, VISUAL MATERIAL AND SIGNS IN LEARNING MATHEMATICS – AN EPISTEMOLOGICAL PERSPECTIVE

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Abstract

Concrete didactical means for learning mathematics, as for instance chips for the number concept, or everyday activities in situations of application as the balance for understanding the algebraic equation, are often seen as an immediate and an effective basis for abstract mathematical knowledge in school mathematics. In this article the specific role of concrete didactical means is analyzed from an epistemological perspective. They are distinguished by two major positions: on the one side they are considered as elements of a *real world of things* and on the other side they are conceived as elements within a *systemic world of mathematical relations*. A major consequence of the following epistemological reflections is that the concrete didactical material in the world of things cannot explain per se mathematical knowledge, especially, when the pedagogical tool (i.e. the concrete material) assists in understanding mathematical knowledge, those elements no longer belong to the physical world, but they belong to the world of mathematical relations.

Key words: epistemology of mathematical knowledge, visualization, manipulatives as mathematical signs, metapher, metonymy

INTRODUCTION

For mathematical knowledge, for teaching and learning, semiotic representations are essential. » … the way students deploy and mobilize ***signs*** (words, letters, etc.) to accomplish mathematical generalization … « (Radford 2001, 1505).

- *Learning mathematics* is about the interpretation and understanding of semiotic means!
- *Teaching mathematics* means for instance among other things, to carefully look at the specific challenges for the learning students when they are asked to interpret mathematical signs and symbols.

How do semiotic means for mathematical knowledge, i.e. mathematical signs and symbols acquire their specific meaning? What could be appropriate and explanatory reference contexts for these signs / symbols? The following epistemological question is fundamental for teaching and learning mathematics: What is the very special kind of mediation, i.e. the »transfer« between signs and objects (*for which the signs stand*) in mathematics?

Object / Reference Context ⟷ Sign / Symbol

Figure 1. Objects and signs

»Epistemology is about the relationship between these types of entities, objects and signs« (Otte 2006, 17). Epistemology of mathematical knowledge offers a *theoretical perspective* on basic requirements of learning and understanding of mathematics.

- Mathematical signs and symbols do not speak immediately for themselves!
- That for what signs and symbols stand, to what they refer and from where they possibly could get explanations, is not fixed permanently but it could suddenly change on the quiet!
- (Young) students should be competent to more consciously perform their interpretations and use of mathematical signs / symbols!

Learning mathematics in school – especially in elementary school – needs manipulatives and visual material, used with the idea to make abstract mathematical knowledge directly accessible in a more or less natural manner. In a fundamental way, mathematical knowledge needs carriers for the knowledge, so called sign vehicles. First of all, mathematical signs or symbols take this role of knowledge carriers. In processes of *learning* mathematics from the beginning concrete manipulatives, visual material and specially prepared concrete representations are used.

The role of manipulatives, of concrete representations and of thing-like didactical means used as an explanatory basis for the understanding and meaningful interpretation will be critically examined with the help of the epistemological triangle (Steinbring 2009).

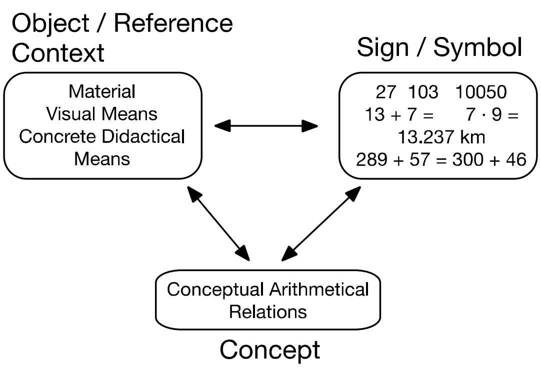


Figure 2. Concrete didactical means and the epistemological triangle

Is it really possible that concrete didactical means can directly offer understanding for abstract signs and symbols (see Fig. 2)?

In the course of the following paragraphs the questions below will be investigated from an epistemological perspective:

- Can manipulatives and visual material directly serve as explanatory reference contexts for partially unknown and questionable signs / symbols?
- Can manipulatives and visual material themselves become partially unknown and questionable signs / symbols for which explanatory reference contexts are needed then?
- Are thing-like or pseudo-thing-like properties of concrete didactical means the distinctive features for making possible the understanding of abstract mathematical knowledge?
- Can a metaphorical background for interpretation using experiences from everyday serve as an

unchanging basis for the understanding of mathematical structures?

These 4 questions are not restricted in the following to exclusively to single paragraphs, but they are dealt with in the whole paper from different perspectives.

CHIPS AND ARITHMETIC – MODIFICATION OF REFERENCE

In early mathematics teaching the counting of things plays a fundamental role for the development of the number concept. Counting objects has to respect the 5 counting principles (cf. Radatz & Schipper 1983). From this perspective, the concrete objects – the chips – take the role of the *object* or of the *reference context* in the epistemological triangle for the number *signs* or number *words* to be explained (see Fig. 3).

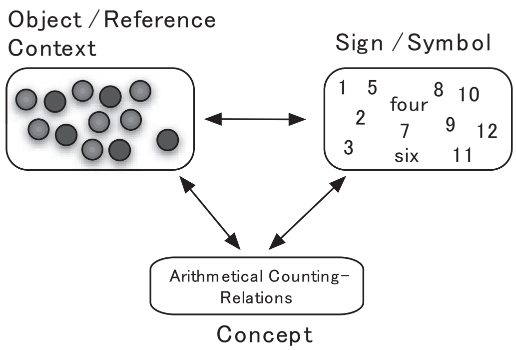


Figure 3. Counting chips and the epistemological triangle

The concrete didactical means, in this case the chips, are used to be the explaining background for the new number signs or number words to be understood. The manipulatives seem to be the certain basis for the new signs or symbols.

As soon as the manipulatives are formed in a pattern structure, the reference of the new signs / symbols to the working material changes (see Fig. 4). The ordering of chips into a rectangle leads to changes in the reference, i.e. in the semiotic mediation between *object / reference context* (the signified) and *sign / symbol* (the signifier). When the 12 chips now ordered in a rectangular shape are no longer simply seen as things to

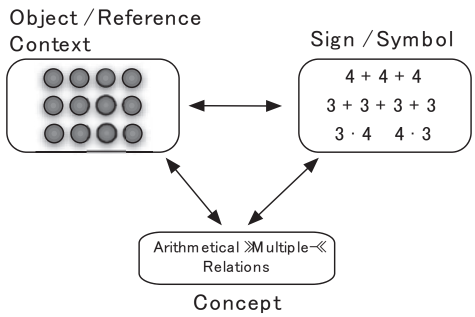


Figure 4. Chips in rectangular form and the epistemological triangle

be counted then the rectangular form needs a specific explanation, before it can be used as an explanatory reference context.

For instance, one has to bring into the chip-configuration an explicit view for »new units«: »one unit of four« and »one unit of three«. Then the combinations »three times four chips« or »four times three chips« could serve as explaining elements of the reference context for the signs »4 + 4 + 4« or »3 · 4« and for »3 + 3 + 3 + 3« or »4 · 3«.

The pseudo-thing-like understanding of chips – expressed in the connection of »one chip represents the number unit 1« – in the counting process of numbers at the beginning now undergoes first changes. The rectangular pattern of chips as a carrier of a structure shows beginning systemic aspects. The rectangle of chips contains first internal relations, which are not determined by thing-like properties but emerge from relations read into the material between the individual chips. In this example the two »sides« of the chip rectangle are interpreted as »new units« and in this way become carriers of relations in a structure.

A further step of a more abstract relational use of the rectangular chip configuration comes up in the case this same pattern shall serve as an explanatory background for the division operation: »12 : 3« und »12 : 4«. Now the whole (of the arranged chips) is subdivided into different part-units. On the side of the arithmetical sign representation the multiplication tasks »3 · 4«, »4 · 3« the division tasks »12 : 3«, »12 : 4« as well as the so called reversed tasks »3 · _ = 12« and »4 · _ = 12« are linked within a systemic connection.

The idea beginning to come up in this example is that concrete didactical material is not able to explain the understanding of abstract mathematical signs and symbols just on the basis of thing-like properties and in a natural way. This would be a very naïve understanding of the role of didactical material. The pseudo-concrete means have to be conceived of as carriers (vehicles) of a systemic structure to be able to contribute to the understanding of mathematical knowledge. This view on didactical material will be further elaborated in more detail using the example of the rectangular 3 · 4 chip configuration.

The 3 · 4 chip configuration can also be put into the position table with the decimal structure (see Fig. 5), and in this way, new systemic relations are introduced for the concrete material.

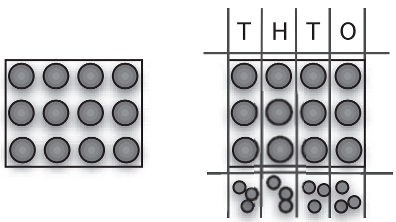


Figure 5. Chips in rectangular form and the position table

The individual chips lose their pseudo thing-like meaning of »one chip corresponds to 1«. The individual chips get their meaning only from the relations they have with the other chips in the system. This system has to obey the rules of the decimal positions. Accordingly, the rectangle of the 12 chips takes over the role of a sign carrier for the threefold sum of the 1111, hence 3333.

And the next step could be to put the 3 · 4 chip configuration – which first seemingly directly represents the arithmetical task »3 · 4« – into the system of the position table multiplication and in this way one can

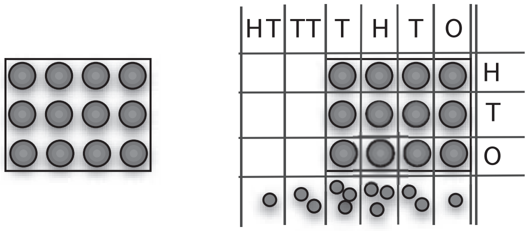


Figure 6. Chips in rectangular form and the multiplicative position table

introduce more complex, new systems relations for these pseudo-concrete material (see Fig. 6).

The meaning of the individual chips is only to be discovered in the system with its relations that obey the rules of decimal positions. Here, the multiplication »1111 · 111 = 123321« is visualized in the chip system (see Fig. 6). For being able to serve as an explanatory reference context, this chip system has to be investigated and explained like a new symbol that is questioned. The thing-like and pseudo thing-like properties of the chips no longer play any role, they can only be understood as elements of a system that explain and define each other in reciprocal ways.

Instead of taking the base 10 also the base (–2) can be chosen. In this way one constructs the so called »negative binary abacus« (cf. Knuth 1969; Gardner 1974; Steinbring 1994). For our example of the 3 · 4 chip configuration a new level of systemic complexity is reached. The epistemological status for the role of sign carriers the chips take over changes. The chips become symbolic marks being able to represent negative numbers in this system of the »negative binary abacus« (see Fig. 7).

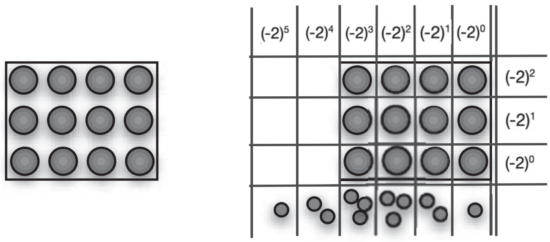


Figure 7. Chips in rectangular form and multiplication in the negative-binary-abacus

The perceptible pattern of the 3 · 4 chip configuration and the produced product show no visible change at all (Fig. 7 compared with Fig. 6). Only the obeying of the new reference for the chips – i.e. symbols that stand for something else – that comes with the variation of the base number, the differences and the new structure can be discovered. In this system with base (–2) the multiplication »(–5) · (+3) = (–15)« is represented. The product again is given by the 3 · 4 chip configuration and the calculated result. In this representational form the calculated sum shows at some position places more than one chip, what is permitted and also productive. A »cleaning of positions« leads to a representation that offers a direct reading of the product »(–15)« (see Fig. 8.)

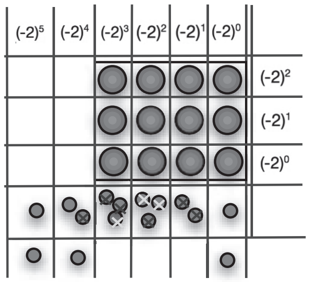


Figure 8. Chips in rectangular form and »(–5) · (+3) = (–15)« in the negative-binary-abacus

For producing the »final form« of the product, the following »cleaning rules« are applied:

(1) Two chips in one column and one chip in the neighbor column left side compensate (are together zero).

(2) Two chips in one column and no chip in the neighbor column left side leads to distributing these two chips on the two columns left side.

Once this systemic structure of the »negative binary abacus« is investigated and becomes more and more familiar, then it can be used as an explanatory reference context for making sense of arithmetical sign-bound questions. The question for an explanation why the product of two negative numbers is positive, can be dealt with the »negative binary abacus« as a possible explaining reference context (see Fig. 9).

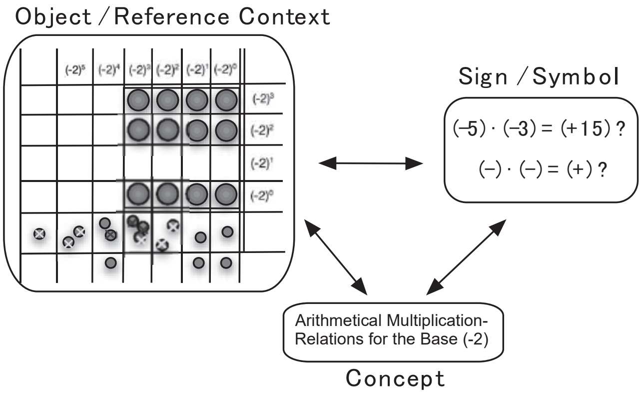


Figure 9. Chips in rectangular form and »(–) · (–) = (+)« in the negative-binary-abacus

The presented examples on the basis of the 3 · 4 chips configuration has clearly shown that the seemingly concrete material of chips change to elements in an abstract system; they are no longer important because of their thing-like properties that are not helpful to provide explanations for mathematical signs / symbols to be understood. The configuration of the 12 chips itself becomes a complex sign / symbol that refers to something else, i.e. to the underlying systemic structure in which the individual chip-elements define each other by internal relationships. The »negative binary abacus« goes far beyond elementary school mathematics, but it clarifies in a particular manner the epistemological perspective taken for didactical material.

The fundamental idea of mathematical entities as elements in a system already plays a role in the sequence of natural numbers. Natural numbers, too, are not based neither on concrete things – as chips, for instance – or thing-like properties. Paul Benacerraf (1984) criticizes an empirical founding of the number concept: „I therefore argue, … that numbers could not be objects at all; for there is no reason to identify any individual number with any one particular object than with any other (not already known to be a number)“ (Benacerraf 1984, 290f.). But if numbers are not things, what else are they? „To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5…… Any object can play the role of 3; that is any object can be the third element in some progression. What is peculiar to 3 is that it defines that role - not being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression“ (Benacerraf 1984, 291).

Young students in primary teaching often tend to identify natural numbers with thing-like properties as can be shown in the example (cf. Steenpaß 2014). The student Sonja has been asked in the course of a clinical interview focusing on the task (see Fig. 10) the following question: „Which of the four little cards fits the best to this number line?“.

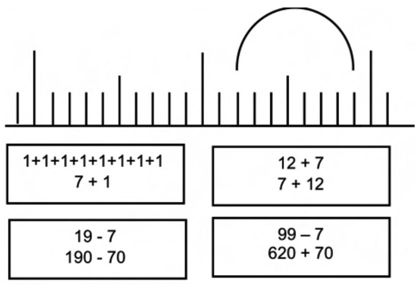


Figure 10. Which card fits the best?

Sonja decides for the task »12 + 7«. Then she shows how she understands this task on the number line. She draws the following circlings and explains them (see Fig. 11). She says the long bars are 100, the medium long bares are 5 or 10, and the small are ones. First she circles the first medium long bar and the two small bars to the right, this makes 12. Then she circles the four small bars left and moreover three small bars around the second long bar, these are together 7.

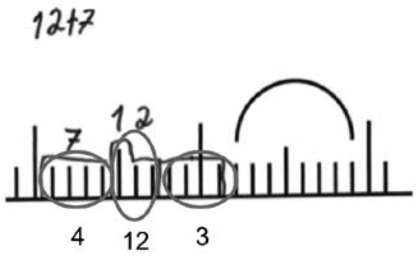


Figure 11. Sonja’s choice: »12 + 7«

Sonja makes a pseudo-thing like interpretation. She uses the bars in a simple way as signs standing for something else, for instance the scaling bars on the number line could refer to concrete wooden bars. These pseudo-things have properties as long, medium long, short or small. For Sonja, these properties are the basis for constructing the numbers 12, 7 and the sum 12 + 7 (cf. Steenpaß 2014; Steenpaß and Steinbring 2013).

This example shows how in the eyes of a young student concrete didactical material can be used as a basis for explaining mathematical, symbolic problems by means of pseudo thing-like characteristics. In contrast to such a view the changed the use and the role of concrete didactical means should be developed and supported from the beginning of learning mathematics. Another example shows that making mathematical structures didactically concrete later are disturbing and from an epistemological point of view inconsistent.

The so called »multiple block systems« (according to Dienes) are a convincing example when these blocks are inserted into the position table (see Fig. 12).

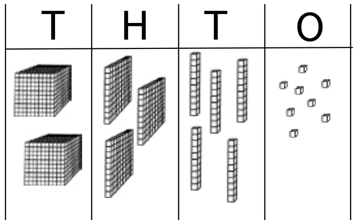


Figure 12. Position table filled with concrete didactical material

From an epistemological point of view the concrete didactical material put into the position table are some kind of re-import of pseudo thing-like properties into a systemic structure. On a simplistic methodological background the concrete things are seen as direct help for the students' understanding and for directly reading off the number looked for. But this mixed system consisting of material means in a systemic structure contains an inconsistence between thing-like and a systemic understanding of the number represented here. Is it the number 2358 (seen from the thing-like properties – then the position table is not needed) or is it 2000Th, 300H, 50T und 8O, leading to 2.030.508 (seen from the systemic structure)?

In many cases the position table is not taken as a productive systemic structure but as a methodological mean to represent numbers and to learn arithmetic. It remains a methodic schema for giving unique terminological attributions to numbers. Accordingly, the following question for the number searched in a test item can only have one and only one correct answer. »You want to represent the number 365 with chips in the position table. How many chips do you need?« (see Fig. 13).

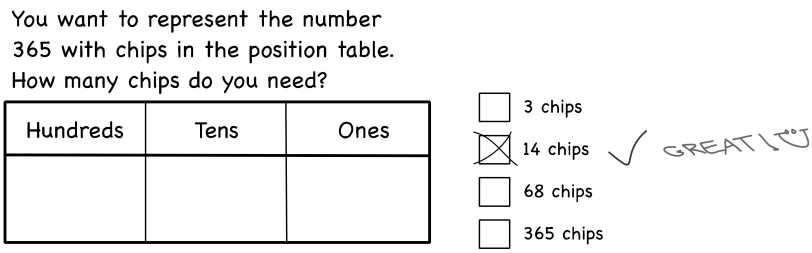


Figure 13. Position table and number of chips for representing the number 365

The expected unique answer here »14 chips« indicates that the position table is mainly used as a formal schema with the guideline that only »one position numbers« are accepted in position columns. A systemic understanding of the position table would allow for many different answers, for instance 365 chips are placed in the One-column, or 68 chips are possible – 33 in the Tenth-column and 35 in the One-column. For investigating and understanding the decimal structure of natural numbers it is more productive to put questions as the following: How many possibilities exist to represent the number 365 with chips in the position table?

Mathematical text books often contain tasks to write natural numbers in position tables in apparent unique and »correct« way (see Fig. 14).

T	H	T	O
0	23	4	0

6 Repairing wrong entries in the position table

a) Tim wanted to write 23 hundreds and 4 tens down in the position table. Write down the number correctly into a position table!

T	H	T	O
	19	7	1

7 Mistakes in the position table?

a) Pina has noted the birth year of her mother in the position table. What is wrong? How does Pina get her result? Which rule for writing numbers into the position table Pina has not taken into account?

Figure 14. Unambiguous representation of numbers in the position table

The position table is used here as a formal protocol for the correct terminology when reading off the numbers as thousands, hundreds, tens and ones. When using the position table as a systemic structure, many possibilities arise for studying manifold relations in the symbolic decimal system. By the way, this is done at many places in the course of developing written arithmetical operations and procedures. The preparation of the written procedure of multiplication is an example for using in a natural way two position numbers as products of positions in the » crosswise position multiplication table«.

In the examples of the »3 · 4 chips« and of the »position table« the epistemological question was asked: Is it possible that thing-like or pseudo thing-like didactical means can be used as explanatory and supporting background or reference context for promoting the learning and understanding of abstract mathematical signs / symbols that need explanation? It has been argued that an initial thing-like understanding of didactical means must change more and more to a systemic understanding of the elements of the didactical means. Not the things themselves nor the thing-like properties serve for understanding mathematical signs / symbols. The insight is needed that the *system* of the elements can promote explanation when the elements are not used by their individual properties but when they exist in the interplay with all other elements of the system.

The reference relation changes from a mediation between thing-like aspects of elements in the reference context and the signs / symbols to a mediation between the systemic relations among the elements in the reference context and the potentially new relations in the sign / symbol system to be understood. In this way it can happen that the systemic contexts of reference themselves can become systems of signs / symbols that need explanation (see Fig. 15).

The philosopher Brian Rotman has described the interplay between mathematical signs / symbols – that what we have noted as mathematical signs – and the references – the thoughts, the signifieds, the ideas – in the following way. „… one recognizes that mathematical signs play a creative rather than a merely descriptive function in mathematical practice. Those things that are ›described‹ – thoughts, signifieds, notions – and the means by which they are described – scribbles – are mutually constitutive: each causes the presence of the other; so that mathematicians at the same time think their scribbles and scribble their thoughts“ (Rotman 2000, 34f.).

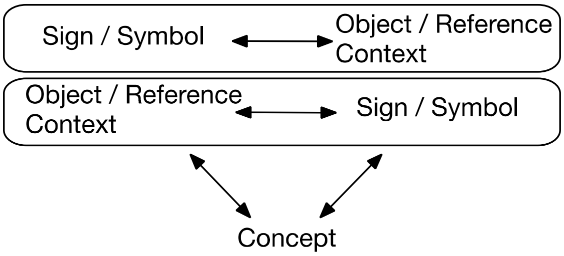


Figure 15. Reciprocal relation between *Object / Reference Context* and *Sign / Symbol*

ARITHMETIC AND ALGEBRA – MODIFICATION OF THE METAPHORIC INTERPRETATION

School algebra is essentially based on arithmetic. When arithmetic is not simply understood as a result oriented calculation but also operative structures are investigated with the help of arithmetical terms, then such a perspective on numbers and structures could positively promote the development of school algebra. Number pyramids are an example for using arithmetical terms in the analysis of arithmetical structures (see Fig. 16).

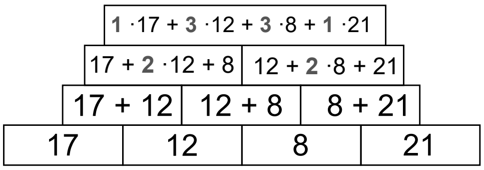


Figure 16. Arithmetical terms and structure of the number pyramid

Operative structures and relations (Wittmann 1985) can be inquired in such arithmetical, substantial learning environments in which not simply results are calculated but the arithmetical terms are progressively taken without first determining the result of the additions. In this example (see Fig. 16) one could for instance find out that the top number of this 4-level pyramid is given as a sum consisting of: one time the left basic corner number, three times the left basic middle number, three times the right basic middle number and one time the right basic corner number.

In this number pyramid with »not calculated results« the numbers in the four basic stones are taken as »general numbers«; they obtain the function of a sort of pre-variable making possible to explore the operative structure and operative connections of the arithmetical problem.

From a general point of view arithmetic is the essential starting point for the introduction and development of school algebra (see Fig. 17).

The explanation of algebraic signs and operations by using the reference context of arithmetic often is accompanied by metaphors that shall give meaning to the new properties of variable and equation. Basic metaphors for variable and for equation are the following two that also are interesting for epistemological questions:

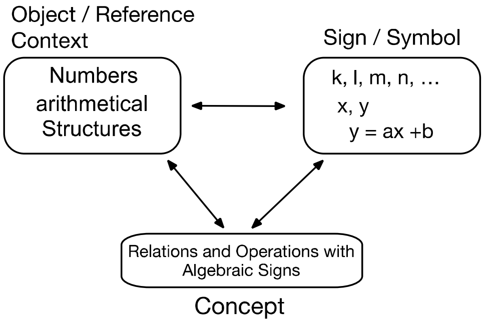


Figure 17. Arithmetical relations as explanatory background for algebraic signs and operations

(1) A variable is a container (containing an unknown number).

(2) An equation is a balance.

A metaphor in the form of »A is a B« transfers a known meaning of B to the still unknown A, or the known B gives the known A another, unusual meaning.

Small bags are an example for pre-variables in the frame of the container metaphor that can be used in elementary mathematics teaching. Task: »Choose a number! Multiply this number with 8, add 4, divide by 2 and then subtract 2! What do you observe?

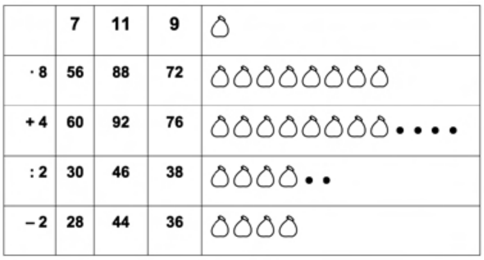


Figure 18. Little bags as »Variable« for the arithmetical relation »4 · Starting-Number«

The small bags play the role of natural pre-variables, by representing an unknown number hidden in the small bag (see Fig. 18, for the container metaphor see for instance Steinweg 2013).

The balance metaphor shall support the understanding of an equation that seems to function in a similar way as a balance (a beam scale). There are two scale pans representing the two sides of the equation and there are objects (»things«) on the two pans, symbolizing the unknown (the variable) and the numbers in the equation (see Fig. 19).

Can the balance be an explanatory reference context for the algebraic equation? Can the metaphor, an equation is a balance, transfer its meaning to the equation (with variables, numbers and the condition of equality) that needs explanation? First one finds similar to the reference between concrete didactical means (chips for instance) and numbers (with arithmetical operations and structures) that also here the given »things« placed on the balance (red cubes and blue balls) are not themselves the elements of the reference context on which the signs / symbols directly refer to. These objects stand for something else, hence they are signs for some other. This symbolic reference has to be actively constructed. In the situation of the balance the red cubes are things to be weighed because their weight is unknown, and this weight should correspond

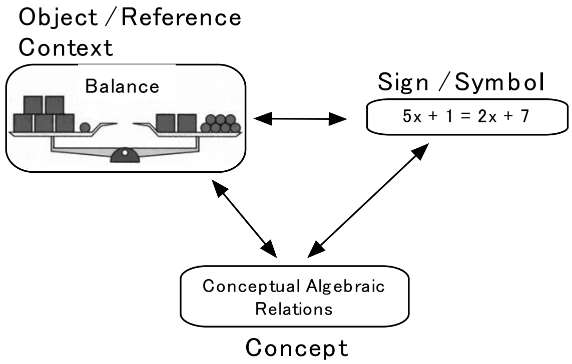


Figure 19. The balance as an explaining reference context for the algebraic equation?

to the unknown variable x. In the situation of the balance the blue balls stand for the weights (1 kg for instance) used to weighed the red cubes, and they should thus correspond to the numbers in the equation. Only through this reinterpretation of the actual balance constellation the weight of an object and units of weight can become elements of a possible reference context for the equation »5x + 1 = 2x + 7«. In the above description of the balance the concrete activity of weighing and the weights have not been explicitly explored.

How convincing and supporting such a concrete balance situation can be for the understanding of an algebraic equation? In the course of a mathematics lesson (cf. Steinbring 2005) the following constellation for two weighting situations is presented to two school students as a basis for understanding a system of two linear equations with two unknowns (see Fig. 20).

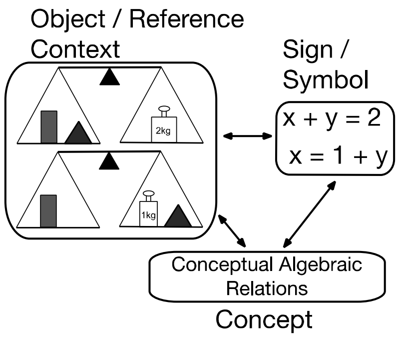
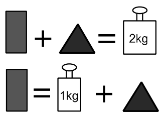


Figure 20. The balance as an explaining reference context for a system of linear equations?

The teacher emphasizes that only concrete and practical activities with the given objects on the balance (a red cylinder and a blue cone) is allowed for »solving the two equations«. Just as being put into a real everyday situation the students should find out the weights of the cylinder and the cone. The metaphor »a system of linear equations with two variables is a twofold representation of a balance« is stated that should make a solution in principle possible by using the concrete balance as a reference context for transferring its meaning to both equations.

In the lesson one can observe how a student quickly finds a solution by using the variables x and y: the weight of the red cylinder is 1.5kg and the weight of the cone is 0.5kg. For making a practical weighing of both objects one would need subdivisions of the used weights (1kg and 2kg). Practical weighing is a process in which the unknown weight of an object is approximated by corresponding fitting weights; starting with a state of imbalance one comes to a state of balance (equilibrium). Are these fundamental ideas of weights and activities of weighing in the process of practical weighing transferrable to the mathematical equations? Does the metaphor »the equation is a balance« function in a naive and direct way?

The teacher in this lesson strongly asks for a »practical solution«; it is only allowed to use the given weights and no other objects than the red cylinder and the blue cone can be chosen. The permitted practical activities are rearrangements of the two concrete objects and the two weights and keeping at the same time the state of equilibrium. But these conditions expressed by the teacher are no original properties of practical weighing. Accordingly the requirement for a proper practical solution in principle cannot be realized. During the classroom interaction a solution is developed – not with x and y, which the teacher has already refused – in which accordingly to the teacher's understanding fulfills the demand for rearranging the objects and the weights: The following connection on both representations of the balance between the fictitious concrete objects and the weights is stated (see Fig. 21):



red + blue = 2
red = 1 + bue

Figure 21. Concrete equation made of things and weights?

The use of the concrete objects (red cylinder, blue cone) and of the weights (1kg and 2 kg) in this way leads to an extensive transformation from the practical, everyday situation with its special rules and activities into the systemic structure of arithmetical and algebraic rules and operations. Despite the fact that the same objects and weights (here visualized) are considered, within the systemics structure their symbolic meaning abruptly changes. The visual pictures of the red and blue object now become signs that refer to unknown corresponding weights, indeed referring only to unknown numbers. The two weights become two known numbers, 1 and 2. At the same time the practical activity of approximate weighting by using a series of different weights is narrowed to (mathematical) activities within equations and under the condition of equilibrium (equality).

The fictitious »practical solution« of the teacher turns out to be a true mathematical elaboration in the system of algebraic relationships. It is in no way the outward appearance of the used objects, weights and signs that make a difference between the practical and the mathematical solution. It is the context used each time with its concepts and operations that makes the difference. In the practical context of weighing, solutions and means and activities for bringing out a solution differ from those in the context of solving algebraic equations. The connection of the visualized concrete objects and weights (in Fig. 21) has the same epistemological status as equations as: »red + blue = 2, red = 1 + blue« or »r + b = 2, r = 1 + b« or »x + y = 2, x = 1 + y«.

Consequently one can state that the metaphor »An equation is a balance« only in a restricted way can transfer meaning from the balance to the equation. The problem is that the concrete situation of practical weighing in everyday and the operative, systemic context of algebraic equations are fundamentally different. The metaphor changes more and more to its conversion in the following way: »A mathematical balance is an

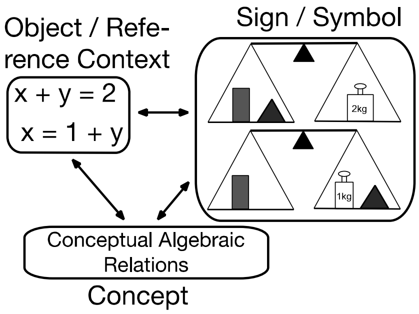


Figure 22. Conversion of the balance metaphor: »A Mathematical Balance is an Equation«

equation« (see Fig. 22).

Objects and activities in situations of practical weighing receive their concrete meaning in the context of everyday praxis with it special rules. In the context of algebraic equations these objects and activities are transformed into a systemic structure and the meaning of the single elements no longer stems from individual properties but is provided by manifold interrelations with the other elements of the system.

In the following problem »Crack the box!« the metaphorical background of a container is used for giving meaning to a variable in an algebraic equation. Black and white boxes – container – are to be filled with matchsticks in way that an equality is reached between the left and the right side of the representation (see Fig. 23).

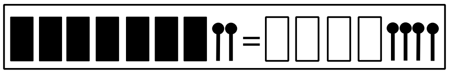


Figure 23. Filling boxes of different color with matchsticks

The task reads as follows:

»Fill the boxes according to the following rules:

1. On both sides of the equal sign same amounts of matchsticks have to be placed.
2. Boxes of the same color contain equal amounts of matchsticks.

Look for different solutions and note them in a table.« (mathbu.ch 7, 2003).

The boxes with an unknown number of sticks represent concrete containers, hence they are metaphors for variables – here for instance for x and y. The inconsistent use of picture like boxes and sticks together with the mathematical equal sign at the same time shall give in this way meaning to the algebraic symbolic equation »7x + 2 = 4y + 4« (see Fig. 24).

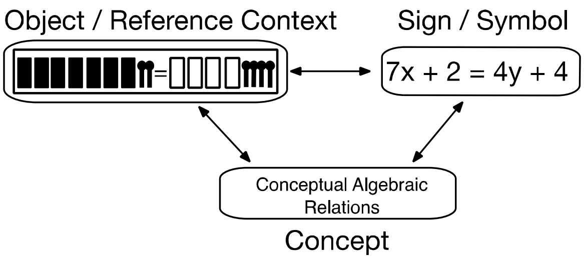


Figure 24. The representation »Crack the Box!« as a metaphor for an algebraic equation

First, here again a correspondence between picture like representations between concrete objects (boxes and matchsticks) and mathematical signs / symbols is aimed at (see Fig. 25).

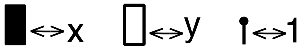
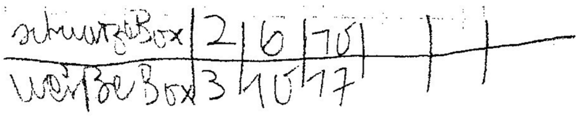


Figure 25. Pictorial representations of real objects and mathematical symbols

In the world of concrete objects, black and white boxes are filled. In the world of signs and symbols one can operate with pictures for boxes and for sticks as with signs according to the rules for solving algebraic equations.

During an empirical study, students (grade 6) have figured out in the course of group work three solutions for this task by means of arithmetical procedures and by trial and error (see Fig. 26).



Black Box	2	6	10		
White Box	3	10	17		

Figure 26. Three solutions for »Crack the Box!«.

On this basis the students have developed the following arithmetical procedure: »Take a number of sticks as a trial for the black box! This number of sticks in the black box is multiplied by 7, and then subtract 2. When the result can be divided by 4, the number of sticks for the black and the white box is found. Otherwise try a new number!«. Example: 6 sticks in the black box multiplied by 7 gives 42, 2 subtracted gives 40; 40 divided by 4 leads to 10. Hence a solution is found: 6 sticks in the black and 10 sticks in the white box.

In this way the students have found solutions for the problem and noted them. They have not used the symbolic notations of the equation – x, y, 2, 4 – but they have pointed at the boxes and have named the sticks and their amounts. They have developed a solution procedure with the help of partially concrete or imitated concrete activities with matchsticks and boxes and with natural numbers. The mathematical work of the students is essentially an arithmetic procedure.

The first three results in the table of the students can be taken as a starting point for investigating connections and structures in relation with the equation »7x + 2 = 4y + 4«. Obviously, the series of numbers in the table can be continued (see Fig. 27).

Black Box	2	6	10	***14***	***18***	***22***			
White Box	3	10	17	***24***	***31***	***38***			

Figure 27. Continuation of solutions for »Crack the Box!«

Is this principle of continuation in general valid? How could we justify it? When we assume that the situation of boxes (see Fig. 28) is not only a pseudo-concrete visualization of concrete objects, but represents symbolic relationships, and can be understood in this form directly as an algebraic equation, then it expresses a still unknown solution (in any case there exit some solutions). The following operative changes then explain the arithmetical structure of the solution (see Fig. 27 and Fig. 29).

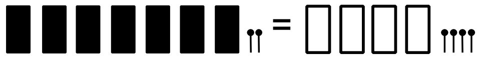


Figure 28. The situation of boxes as algebraic equality with signs for variables and numbers

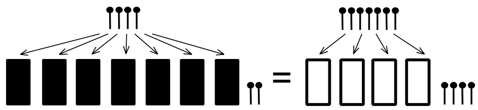


Figure 29. Operative changes on both sides of the equation

Independent of what (unknown) solution is given in the situation of boxes (Fig. 28), when on the left side always 4 sticks are added to each of the 7 black boxes and at the same time on the right side always 7 sticks are added to each of the 4 black boxes then equality is maintained. 7 times 4 sticks have been added on the left and 7 times 4 sticks have been added on the right side. The solution set is given in the following symbolic form:

$$L = \{(-2 + 4n\ ,\ -4 + 7n)\ /\ n = 1, 2, 3, 4, \ldots\ldots\}$$

This diagrammatic representation (Fig. 29) does not lead to a general solution of the diophantine equation, but can it be understood as a kind of action proof that demonstrates that the series of numbers in the table in Fig. 27 can be continued.

When investigating the question in which way boxes and matchsticks can be seen as productive metaphors – boxes as containers, sticks as natural numbers – that are able to transfer their meaning to algebraic equations, we are again facing the epistemological problem that visual pictures of concrete objects with their practical activities have been transformed into mathematical signs and symbols and now they must obey the rules of the algebraic system. Again a reciprocal interplay is constituted between the signs / symbols that are questioned and that have to be explained and the reference context as a basis for (partial) explanations. The mediation between reference context and sign / symbol is subject to changes of their roles dependent of actual interpretations and questions (see Fig. 30).

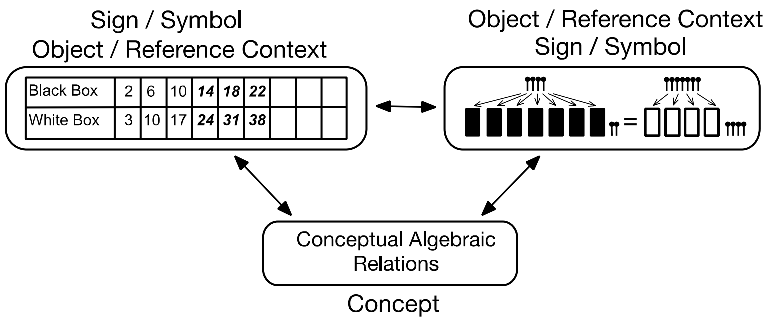


Figure 30. Exchange of explaining Reference Context and Sign / Symbol to be explained

In the development of school algebra the metaphors of the balance and of the container are used to transfer meaning from a concrete or a pseudo-concrete situation with its praxis of activities and interpretations to the algebraic equation and to the variable. In the course of intended transfer of meaning to algebraic concepts, the visual representations of the concrete didactical means (little bags, cylinder, cone, weights, boxes, matchsticks, etc.) themselves become signs / algebraic symbols and they have now to be interpreted and used according to the systemic rules for solving algebraic equations (in school algebra) (see Fig. 31).

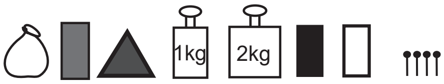


Figure 31. Visualized concrete didactical means transferred to algebraic symbols x, y, 1, 2, 4.

The iconographic representation of these »visual symbols« are not conform to the usual way of writing mathematical signs / symbols. They rather look like pictograms and have not the form of mathematical inscriptions. By the change from a pseudo-real world of concrete means into the world of algebraic structure with its systemic relationships they now function in the same way as mathematical signs / symbols do. The metaphor of everyday changes to a metaphor of algebraic equality.

EPISTEMOLOGICAL RÉSUMÉ

The concrete didactical means – chips, little bags, objects, weights, boxes, sticks, etc. – come from a world of things with concrete properties and practical activities in this praxis. When these concrete didactical means are used to provide explanations for abstract mathematical symbols, at once their epistemological status changes according to the new conditions. They no longer belong to the world of things but they have entered the systemic world of mathematical relations (see Fig. 32). As elements of the systemic world, the formerly concrete means do no longer have individual properties, that define them, now they are determined by the manifold of reciprocal relations with other elements of the systemic world of signs and symbols. The visual representations of concrete didactical means in principle themselves mutate to signs / symbols.

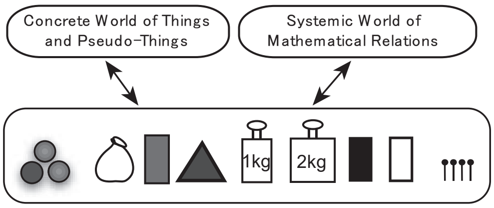


Figure 32. Visualized concrete didactical means: Elements of separate »World«.

The transformation of didactical means from things to elements in a system requires the change of the background of meaning. The chips no longer keep their quasi empirical status of discrete things that primarily serve for counting objects. And the »unknown number« is not the »unknown« content for a variable. The background of meaning in the world of things – things for counting, the container metaphor, the balance metaphor – changes with the requirement to provide explanations for symbolic mathematical connections. The variable becomes a symbolic entity in the system of relations in the algebraic praxis of solving equations. The metaphor »The equation is a balance« is contrasted by the metaphor »The *mathematical* balance is an equation«.

The didactical means are either elements of the real world of things or elements of the systemic world of mathematical relations. In the course of learning arithmetic and algebra in school, which is essentially based on concrete material, the problem arises, how students' understanding can develop and can change when concrete and thing-like didactical means switch to systemic, theoretical signs. When discussing the meaning of fundamental mathematical concepts – the number concept, the variable, the equation – students are mostly reminded to look at the concrete material and its function: chips as markers to count, numbers as an »unknown content of variables« and the equation as being a balance, performing the same activity on both sides. This view might lead to an overemphasis of a procedural work in arithmetic and algebra.

The epistemological problem discussed that elementary mathematics – school arithmetic and algebra – cannot be directly linked with concrete didactical means in the sense that concrete didactical material in the world of things cannot explain per se mathematical knowledge, is also a problem for the application of mathematics – for instance when working on word problems or with modelling activities in school mathematics (cf. Voigt 2013). The connection between mathematics and an application situation is not seen as a direct translation, but as a structural extension. „The relation between real world facts and mathematics is not constituted by *neglect of sufficient many realistic details for the preparation of a translation*, but by a *theoretical change of the empirical facts to a structural extension of the real world facts*, …" (Schwarzkopf 2007, 104).

In his famous speech »Geometry and Experience« Einstein (1921) has characterized the difference between mathematical theory and reality with the following statement: „… as far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." (Einstein 1921, 124). Analogous to this statement one could say for the meaning of didactical means and for the understanding of mathematical knowledge: „… as far as the didactical means are elements of a concrete world of things they are not supportive for the understanding of mathematical knowledge; and as far as they support the understanding of mathematical knowledge they are no longer elements of the concrete world of things."

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